Optimization Starter Pack Solutions

1. If the length is x and the width is y we have:

$$Max \;\; A = xy \quad s.t. \quad 2x + 2y = 24$$

From constraint: $2x + 2y = 24 \rightarrow x + y = 12 \rightarrow y = 12 - x$

Now:

$$A = xy = x(12 - x) = 12x - x^2 \rightarrow A' = 12 - 2x = 0 \rightarrow x = 6\,m$$

Plugging back to the rearranged constraint equation:

$$y = 12 - x = 12 - 6 \rightarrow y = 6\,m$$

So, maximum area is:

$$A = xy = (6)(6) = 36\,m^2$$

2. If the radius is r and the height is h we have:

$$Min \;\; A = \pi r^2 + 2\pi rh \quad s.t. \quad \pi r^2 h = 27\pi$$

From constraint: $\pi r^2 h = 27\pi \rightarrow r^2 h = 27 \rightarrow h = \frac{27}{r^2}$

Now:

$$A = \pi r^2 + 2\pi rh = \pi r^2 + 2\pi r\left(\frac{27}{r^2}\right) = \pi\left(r^2 + \frac{54}{r}\right) \rightarrow$$

$$A' = \pi\left(2r - \frac{54}{r^2}\right) = 0 \rightarrow 2r - \frac{54}{r^2} = 0 \rightarrow \; 2r = \frac{54}{r^2} \rightarrow \; 2r^3 = 54 \rightarrow \; r^3 = 27 \rightarrow r = 3\,ft$$

Plugging back to the rearranged constraint equation:

$$h = \frac{27}{r^2} = \frac{27}{3^2} = \frac{27}{9} = 3ft$$

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3. $Area = \frac{1}{2}[AB\ .\ PB]$

Now, assuming $OB = x$ [since B has coordinates of (x,0)], then $AB = AO + OB = 4 + x$

PB is the y coordinate of point P. Therefore $PB = \sqrt{16 - x^2}$

So,

$$A = \frac{1}{2}\left[(4 + x)\left(\sqrt{16 - x^2}\right)\right]$$

The objective is already in terms of one variable (because we incorporated the constraint that point P(x,y) is on the curve and therefore x and y relationship is governed by $y = \sqrt{16 - x^2}$).

Now we derive it for maximization:

$$A' = \frac{1}{2}\left[(1)\left(\sqrt{16 - x^2}\right) + (4 + x)\left(\frac{1}{2\sqrt{16 - x^2}}\right)(-2x)\right]$$

$$= \frac{1}{2}\left[\left(\sqrt{16 - x^2}\right) + (4 + x)\left(\frac{-x}{\sqrt{16 - x^2}}\right)\right] = \frac{1}{2}\left[\left(\sqrt{16 - x^2}\right) + \left(\frac{-4x - x^2}{\sqrt{16 - x^2}}\right)\right] = 0 \rightarrow$$

$$\frac{16 - x^2 + (-4x - x^2)}{2\sqrt{16 - x^2}} = 0 \rightarrow\ 16 - x^2 + (-4x - x^2) = 0 \rightarrow 2x^2 + 4x - 16 = 0$$

$$\rightarrow\ x^2 + 2x - 8 = 0 \rightarrow (x + 4)(x - 2) = 0 \rightarrow x = -4\ or\ x = +2$$

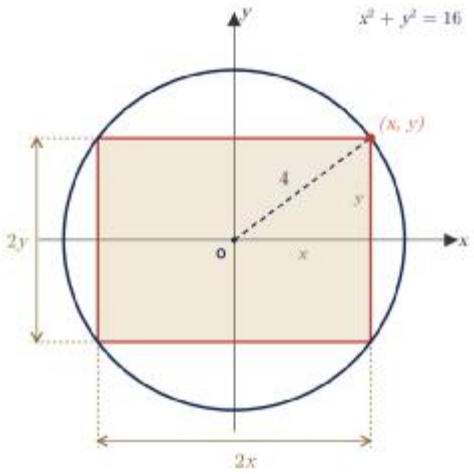
Since x is in the first quadrant x=2 is admissible (note that x=-4 gives the triangle with "smallest" area which is zero! But the largest area is given when x=+2)

$$x = 2 \rightarrow y = \sqrt{16 - x^2} = \sqrt{16 - 2^2} = \sqrt{12}$$

So, point P is $P\left(2, \sqrt{12}\right)$

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4.



$$Max \;\; A = (2x)(2y) = 4xy \quad s.t. \quad x^2 + y^2 = 16$$

From constraint: $x^2 + y^2 = 16 \rightarrow y^2 = 16 - x^2 \rightarrow y = \sqrt{16 - x^2}$
Now:

$$A = 4xy = 4x\sqrt{16 - x^2} \rightarrow$$

$$A' = 4\left[(1)\left(\sqrt{16 - x^2}\right) + (x)\left(\frac{1}{2\sqrt{16 - x^2}}\right)(-2x)\right] = 4\left[\left(\sqrt{16 - x^2}\right) + x\left(\frac{1}{\sqrt{16 - x^2}}\right)(-x)\right]$$

$$= 4\left[\left(\sqrt{16 - x^2}\right) + \left(\frac{-x^2}{\sqrt{16 - x^2}}\right)\right] = 0 \rightarrow$$

$$4\left[\frac{16 - x^2 + (-x^2)}{\sqrt{16 - x^2}}\right] = 0 \rightarrow \; 16 - 2x^2 = 0 \rightarrow \; 2x^2 = 16 \rightarrow \; x^2 = 8 \rightarrow x = \sqrt{8}$$

Plugging back to the rearranged constraint equation:

$$y = \sqrt{16 - x^2} = \sqrt{16 - (\sqrt{8})^2} = \sqrt{16 - 8} \rightarrow y = \sqrt{8}\, m$$

So, maximum area is:

$$A = 4xy = 4\left(\sqrt{8}\right)\left(\sqrt{8}\right) = 32\, m^2$$

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5.

$$Max \;\; R = (p)(q) \quad s.t. \quad 3p^2 + q^2 = 72$$

From constraint: $3p^2 + q^2 = 72 \rightarrow q^2 = 72 - 3p^2 \rightarrow q = \sqrt{72 - 3p^2}$
Now:

$$R = pq = p\sqrt{72 - 3p^2} \rightarrow$$

$$A' = \left[(1)\left(\sqrt{72 - 3p^2}\right) + (p)\left(\frac{1}{2\sqrt{72 - 3p^2}}\right)(-6p)\right]$$

$$= \left[\left(\sqrt{72 - 3p^2}\right) + p\left(\frac{1}{\sqrt{72 - 3p^2}}\right)(-3p)\right]$$

$$= \left(\sqrt{72 - 3p^2}\right) + \left(\frac{-3p^2}{\sqrt{72 - 3p^2}}\right) = 0 \rightarrow$$

$$\frac{72 - 3p^2 + (-3p^2)}{\sqrt{72 - 3p^2}} = 0 \rightarrow \; 72 - 6p^2 = 0 \rightarrow \; 6p^2 = 72 \rightarrow \; p^2 = 12 \rightarrow p = \sqrt{12}\; dollars$$

Plugging back to the rearranged constraint equation:

$$q = \sqrt{72 - 3p^2} = \sqrt{72 - 3(\sqrt{12})^2} = \sqrt{72 - 3(12)} = \sqrt{36} = 6\; pizzas$$

So, maximum revenue is:

$$R = pq = \left(\sqrt{12}\right)(6) = 6\sqrt{12}\; dollars$$

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